Going down Theorem (Assume rings are Noetherian)

Systems of parameters and the PIT give us the following result about algebras over local rings:

Prop: Let $(R, \mathfrak{m})$ be a local ring and $\varphi: R \to S$ with $\mathfrak{m}S \neq S$. Then $\operatorname{codim} \mathfrak{m}S \leq \operatorname{codim} \mathfrak{m}$.

Pf: First note that $\mathfrak{m} \subseteq \varphi^{-1}(\mathfrak{m}S) \neq S$, so $\varphi^{-1}(\mathfrak{m}S) = \mathfrak{m}$.

Let $x_1, \dots, x_d \in \mathfrak{m}$ be a system of parameters for R and set $I = (x_1, \dots, x_d)S$. Let P be a minimal prime over $\mathfrak{m}S$ and suppose

$$I \subseteq Q \subsetneq P$$

where Q is prime. Then

$$(x_1, \dots, x_d) \subseteq \varphi^{-1}(I) \subseteq \varphi^{-1}(Q) \subseteq \varphi^{-1}(P) = \mathfrak{m}$$

$\mathfrak{m}$ is minimal over $(x_1, \dots, x_d)$, so $\varphi^{-1}(Q) = \mathfrak{m}$. Thus, $\mathfrak{m}S = \varphi^{-1}(Q)S \subseteq Q$ so $P = Q$. Thus P must be minimal over I, which is generated by d elements. Thus

$$\operatorname{codim} \mathfrak{m}S \underset{\substack{\uparrow \\ \text{PIT}}}{\leq} d = \operatorname{codim} \mathfrak{m}. \quad \square$$

We can extend this to a result about the dimension of <u>local</u> R-algebras:

Theorem: Let $\varphi: (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a map of local rings s.t. $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. Then

$$\dim S \leq \dim R + \dim S/\mathfrak{m}S.$$

Pf: Let $x_1, \ldots, x_d \in \mathfrak{m}$ be a system of parameters for R and $y_1, \ldots, y_e \in \mathfrak{n}$ such that their images in $S/\mathfrak{m}S$ form a system of parameters.

In particular, $d = \dim R$, $e = \dim S/\mathfrak{m}S$.

Then the image $\bar{\mathfrak{n}}$ of $\mathfrak{n}$ in $S/\mathfrak{m}S$ is the max'l ideal, so

$$\bar{\mathfrak{n}}^{\alpha} \subseteq \overline{(y_1, \ldots, y_e)} \qquad \text{for } \alpha \gg 0$$

$$\Rightarrow \quad \mathfrak{n}^{\alpha} \subseteq (y_1, \ldots, y_e) + \mathfrak{m}S$$

and for $\beta \gg 0$, $\mathfrak{m}^{\beta} \subseteq (x_1, \ldots, x_d)$. Thus

$$\begin{aligned} \mathfrak{n}^{\alpha\beta} &\subseteq ((y_1, \ldots, y_e) + \mathfrak{m}S)^{\beta} \\ &\subseteq (y_1, \ldots, y_e) + \mathfrak{m}^{\beta} S \\ &\subseteq (y_1, \ldots, y_e, x_1, \ldots, x_d)S, \end{aligned}$$ so

$\mathfrak{n}$ is minimal over. Thus the PIT says

$$\dim S \leq d + e. \quad \square$$

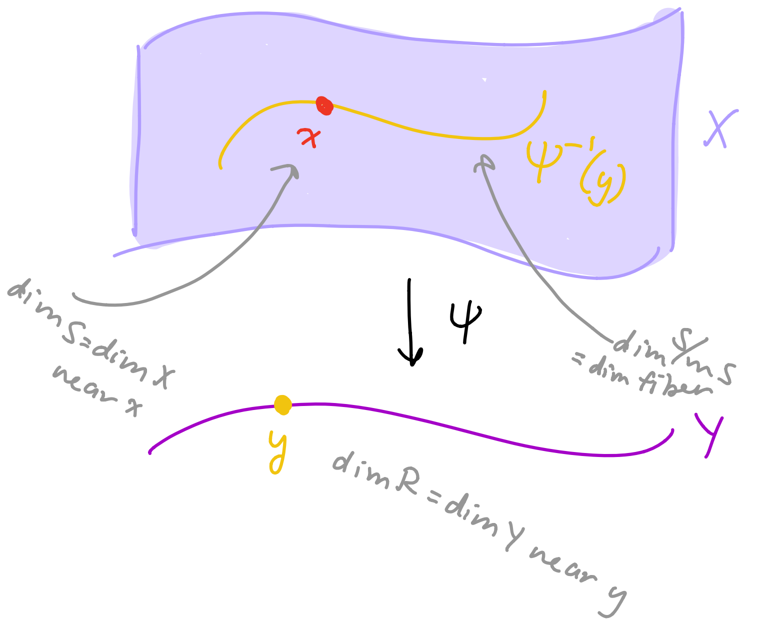
What is this saying geometrically? Note that

$$S/\mathfrak{m}S = R/\mathfrak{m} \otimes_R S = k(\mathfrak{m}) \otimes_R S,$$

so $\operatorname{Spec}(S/\mathfrak{m}S)$ is the fiber of the corresponding Spec map.

Roughly,
if $\varphi : X \to Y$ is a map of (sufficiently nice) varieties or schemes $\varphi(x) = y$ w/ S and R the corr. local rings, respectively, then $\dim R = \operatorname{codim} y$, $\dim S = \operatorname{codim} x$, $\dim S/\mathfrak{m}S = \dim \varphi^{-1}(y)$.

In particular, if x and y are closed points, then we have the following picture.



In this situation,

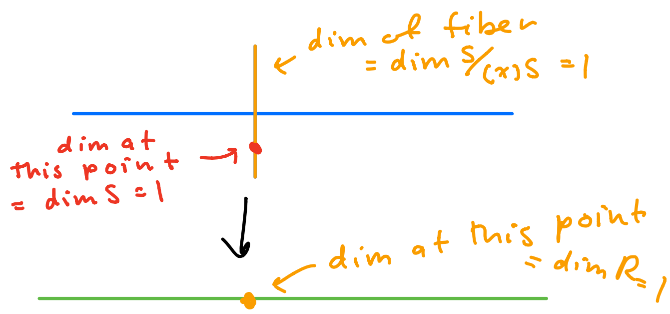
$$\dim X \leq \dim Y + \dim \varphi^{-1}(y)$$

Note that equality doesn't always hold:

Ex: Define $\varphi : \mathbb{C}[x] \to \mathbb{C}[x,y]/(x(y-1))$ and consider the induced map on local rings

$$\underset{\substack{\parallel \\ R}}{\mathbb{C}[x]_{(x)}} \longrightarrow \underset{\substack{\parallel \\ S}}{\left(\mathbb{C}[x,y]/(x(y-1))\right)_{(x,y)}} \cong \left(\mathbb{C}[x,y]_{(x,y)}/(x)\right) \cong \mathbb{C}[y]_{(y)}$$

So $\underset{\substack{\parallel \\ 1}}{\dim S} < \underset{\substack{\parallel \\ 1}}{\dim R} + \underset{\substack{\parallel \\ 1}}{\dim S/(x)S}$



There are even irreducible examples where the inequality is strict (e.g. a surface blown up at a point).

For flat R-algebras (and integral extensions — which we won't prove), equality holds. To prove this in the flat case, we need the following:

Lemma: Let $\varphi : R \to S$ s.t. S is flat over R. If $P' \subseteq P \subseteq R$ are prime ideals and $Q \subsetneq S$ prime s.t. $\varphi^{-1}(Q) = P$, then there is a prime $Q' \subseteq Q$ s.t. $\varphi^{-1}(Q') = P'$. Moreover, Q' can be taken to be any prime contained in Q and minimal over $P'S$.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \cup & & \cup \\ P & \text{———} & Q \\ \cup & & \cup \\ P' & \text{———} & Q' \end{array}$$

This immediately implies the classical going down theorem:

Going down Theorem (for flat extensions): $\varphi : R \to S$ as in lemma. If $P_0 \supset P_1 \supset \dots \supset P_n$ is a chain of prime ideals in R and $Q_0 \subsetneq S$ prime such that $\varphi^{-1}(Q_0) = P_0$, then there's a chain of primes $Q_0 \supset Q_1 \supset \dots \supset Q_n$ in S with $\varphi^{-1}(Q_i) = P_i$.

Thus, we just need to prove the lemma.

Pf: $P' \subseteq P = \varphi^{-1}(Q)$, so $P'S \subseteq \varphi^{-1}(Q)S \subseteq Q$.

Let $Q' \subseteq Q$ be a prime minimal over $P'S$.

We want to replace $R \to S$ with $R/P' \to S/P'S$ and assume $P' = 0$. First we need that $S/P'S$ is flat over R/P':

Let $M' \hookrightarrow M$ R/P'-modules. Then they are R-modules, so by flatness of S, we have

$$
\begin{array}{ccc}
S \otimes_R M' & \hookrightarrow & S \otimes_R M \\
\| & & \| \\
S \otimes_R (R/P' \otimes_{R/P'} M') & \hookrightarrow & S \otimes_R (R/P' \otimes_{R/P'} M) \\
\| & & \| \\
\underbrace{(S \otimes_R R/P')}_{= S/P'S} \otimes_{R/P'} M' & \hookrightarrow & \underbrace{(S \otimes_R R/P')}_{= S/P'S} \otimes_{R/P'} M
\end{array}
$$

so $S/P'S$ is flat over R/P', so assume $P' = 0$. Thus, we want to show $\varphi^{-1}(Q') = 0$.

Since S is flat over R, it is torsion-free over R. Since R is an integral domain (0 is prime), this means all elements of R are NZD on S.

Q' is a minimal prime of S, so it's an associated prime of $0 \subseteq S$. Thus $Q' - \{0\}$ consists of zero divisors on S, so $\varphi^{-1}(Q') = 0$. $\square$

Now we can show that equality in the theorem holds for flat algebras:

Cor: Let $\varphi : (R, m) \to (S, n)$ be a map of local rings such that the image of m is in n and S is flat over R. Then

$$\dim S = \dim R + \dim S/mS.$$

Pf: We already showed one inequality, so we just need $\dim S \geq \dim R + \dim S/mS$.

Let $Q \subseteq S$ be a prime minimal over mS s.t.

$$\dim S/Q = \dim S/mS.$$

Then $\dim S \geq \dim S/Q + \operatorname{codim} Q = \dim S/mS + \operatorname{codim} Q$.

Thus, it suffices to show $\operatorname{codim} Q \geq \dim R$.

Q contains mS, so $\varphi^{-1}(Q) = m$. Let $m \supsetneq P_1 \supsetneq \cdots \supsetneq P_d$ be a max'l length chain s.t. $d = \dim R$.

Going down says $\exists \ Q \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_d$ s.t. $\varphi^{-1}(Q_i) = P_i$.
Thus, $\operatorname{codim} Q \geq \dim R$. $\square$

We can now finally calculate the dimension of a polynomial ring:

Thm: If R is a ring, then $\dim R[x] = 1 + \dim R$. In particular, if k is a field, $\dim k[x_1, \dots, x_r] = r$.

Pf: The second statement follows from the first by induction on r.

For the first statement, let $P_0 \subsetneq \cdots \subsetneq P_d$ a chain of prime ideals in R. Then $P_i R[x]$ is the polynomials w/ coefficients in P_i. Thus, we have

$$P_0 R[x] \subsetneq \cdots \subsetneq P_d R[x] \subsetneq \underbrace{P_d R[x] + (x)}$$

$$R[x] / P_d R[x] + (x) \cong \left(R/P_d \, [x] \right) / (x) \cong R/P_d$$

a chain of primes in $R[x]$. Thus, $\dim R[x] \geq \dim R + 1$.

For the other inequality, note that

$$\dim R[x] = \sup \{ \operatorname{codim} m \mid m \subseteq R[x] \text{ maximal} \}$$

Thus, it suffices to show that for max'l $\mathfrak{m} \subseteq R[x]$,

$$\operatorname{codim} \mathfrak{m} \le \dim R + 1.$$

Let $P = \mathfrak{m} \cap R$. Then the map $R_P \to R[x]_{\mathfrak{m}}$ sends PR_P into $\mathfrak{m}R[x]_{\mathfrak{m}}$, so by the theorem at the beginning of the section,

$$\dim R[x]_{\mathfrak{m}} \le \underbrace{\dim R_P}_{\le \dim R} + \underbrace{\dim R[x]_{\mathfrak{m}}/PR[x]_{\mathfrak{m}}}_{\text{want this to be } \le 1}$$

Note that since $R \cap \mathfrak{m} = P$, we have $R[x]_{\mathfrak{m}} = (R_P[x])_{\mathfrak{m}}$.

Thus,

$$R[x]_{\mathfrak{m}}/PR[x]_{\mathfrak{m}} = (R_P[x])_{\mathfrak{m}}/(PR_P[x])_{\mathfrak{m}}$$

$$= \left(R_P/PR_P\right)[x]_{\mathfrak{m}}.$$

R_P/PR_P is a field, so $\left(R_P/PR_P\right)[x]$ is a PID, so the image of $\mathfrak{m}$ in it is principal. Thus, the PIT says

$$\dim \left(R_P/PR_P\right)[x]_{\mathfrak{m}} \le 1,$$

and we're done. $\square$